NORTH-HOLLAND

A Constrained Least-Squares Approach to the Rapid Reanalysis of Structures

Ho-Jong Jang
Department of Mathematics
Hanyang University
Seoul 133-791, Korea

Submitted by Richard A. Brualdi

ABSTRACT

A procedure to reanalyze a damaged structure using a finite-element force method of analysis is presented. Perturbation analysis of constrained least-squares problems is adapted to handle reanalysis by the force method, and related theoretical and numerical results are presented. © 1997 Elsevier Science Inc.

1. INTRODUCTION

Given the external loads on a structure, the object of structural analysis is to determine the resulting internal forces, stresses, and displacements. The solution to this problem is provided by a variational principle (minimization of energy) subject to the linear elastic relationships among the nodes and elements of the finite-element model of the structure, which can be stated as the quadratic programming problem

$$\operatorname*{Min}_{f} \ \tfrac{1}{2} f^T A f \qquad \text{subject to} \quad Ef = s. \tag{1}$$

This work was partially supported by the Hanyang University Research Grants in 1995.
E-mail: jang@fermat.hanyang.ac.kr

LINEAR ALGEBRA AND ITS APPLICATIONS 265:185–202 (1997)

655 Avenue of the Americas, New York, NY 10010
0024-3795/97/$17.00
PII S0024-3795(96)00602-7

A similar minimum principle can be stated in terms of displacements rather than forces. First-order necessary conditions for a solution to the quadratic programming problem above are given by the 2×2 block system of linear equations

$$\begin{bmatrix} A & E^T \\ E & 0 \end{bmatrix} \begin{bmatrix} f \\ \lambda \end{bmatrix} = \begin{bmatrix} 0 \\ s \end{bmatrix}, \tag{2}$$

where λ is a vector of Lagrange multipliers. Here A is the element flexibility matrix (or equivalently the element stiffness matrix is A^{-1}; A and A^{-1} have block-diagonal structures), E is the equilibrium matrix, s is the vector of external loads, $-\lambda$ is the displacement vector, and f is the system force vector. We assume that A is a symmetric positive definite matrix of order n and that E is an $m \times n$ matrix of rank m. There are two methods generally used to calculate (1) or (2), the displacement method and the force method.

The purpose of a reanalysis procedure is to analyze a damaged structure using, as much as possible, quantities calculated in the analysis of the original structure. Various means to accomplish reanalysis of damaged structures have been investigated in [2, 17, 10, 18]. This work, for the most part, has been based on the matrix displacement method and iterative schemes, because the displacement method is easier to implement on digital computers, especially for large sparse systems, and makes use of well-established techniques of numerical linear algebra. However, the force method is sometimes preferable for reanalysis because it utilizes a portion of earlier computations in order to solve such modified problems without starting the computations over from the beginning. Unfortunately, most implementations of the force method have suffered from excessive fill or numerical difficulties, or both. A series of papers [9, 5, 8, 6, 7, 16] have given an implementation of the force method which is numerically stable and preserves sparsity for large scale problems, and some reanalysis work has been done using the force method [1]. Recent research [3, 4, 16] indicates that the force method is a viable alternative not only to the solution of problems of dynamics and weight optimization but also to reanalysis.

Past investigators have defined damage models by removing structural finite elements entirely or reducing values of the design parameters which affect the structure's flexibility and mass. From these studies, it can be concluded that two damage models can be defined, namely *large scale* and *small scale*. The large-scale damage consists in complete removal of finite elements due to ballistic damage, for instance. The small-scale damage consists in the reduction of finite-element properties so that flexibility is increased. This type of damage could represent increased flexibility due to fatigue cracks or small holes caused by ballistic impacts.

In this paper we discuss the reanalysis based on the force method in the case of small-scale damage. This case also has two subcases. The first is when either one or two elements have been modified, and the second is when almost all of the components of the stiffness matrix have been modified. Both subcases will be discussed in the next two sections.

The force method, as derived in [11], is now summarized.

FORCE METHOD.

Step 1. Find a particular solution f_p to $Ef = s$:

$$Ef_p = s. \tag{3}$$

Step 2. Find the self-stress matrix N such that $EN = 0$, and solve

$$N^TANf_0 = -N^TAf_p, \qquad f_0 \text{ a redundant force vector.} \tag{4}$$

Step 3. Set $f = f_p + Nf_0$.

2. REANALYSIS WITH QR FACTORIZATION

Given a particular solution f_p in (3), the main task of the force method is the computation of the redundant force vector f_0 which satisfies (4). The system (4) is simply the normal equation for the weighted least-squares problem:

$$\operatorname*{Min}_{f_0} \left\| G^{-1}(Nf_0 + f_p) \right\|_2, \tag{5}$$

where N is an $n \times (n - m)$ matrix and G is the Cholesky factor of the element stiffness matrix A^{-1}. The traditional method of normal equations consists in the direct application of Cholesky's method to the symmetric positive definite matrix N^TAN. Unfortunately, explicitly forming the matrix N^TAN can lead to loss of sparse structure of N and worsening of the conditioning of the problem. A better approach in this regard is to apply orthogonal transformations to the matrix $G^{-1}N$, leading to an algorithm of the following form:

ORTHOGONAL FACTORIZATION.

$$P_1 G^{-1} N P_2^T = Q\begin{bmatrix} R \\ 0 \end{bmatrix},$$

$$\begin{bmatrix} c \\ d \end{bmatrix} = -Q^T P_1 G^{-1} f_p, \tag{6}$$

$$f_0 = P_2^T R^{-1} c,$$

where R is an upper triangular matrix of order $n - m$, P_1 and P_2 are permutation matrices of order n and $n - m$, respectively, Q is an orthogonal matrix of order n, and c and d are vectors of length $n - m$ and m, respectively.

Several methods for solving problems of the form (5) are described in [13].

Reanalysis by the force method based on QR factorization has been done in [16] for the case that only one element has been modified. The element flexibility matrix $A = \text{diag}[A_k]$, where each A_k is an $n_k \times n_k$ symmetric positive definite, is symmetric positive definite. If we assume only one block of the matrix A has been modified, that is, one element changed, then A will be modified by changing one A_k to $A_k + \delta_k \delta_k^T$ where each δ_k is $n_k \times n_k$. Suppose we use an orthogonal factorization to solve (5); then the advantage of the force method is that one can use the QR factorization of the unperturbed problem of (6) to solve the perturbed problem

$$N^T\left(A + e_k \delta_k \delta_k^T e_k^T\right)N(f_0 + \Delta f_0) = -N^T\left(A + e_k \delta_k \delta_k^T e_k^T\right)f_p, \tag{7}$$

where each e_k is an $n \times n_k$ matrix having all zero components except the kth block, which is the $n_k \times n_k$ identity. Based on the assumption that only one block of the matrix A has been modified, Plemmons and White [16] established following theorem.

THEOREM 1. *Let f_0 be the solution of* (4). *Let $G^{-1}N = QR$, where G is a Cholesky factor of symmetric positive definite matrix A^{-1} and N has full* *column rank. Then the solution of* (7) *is given by* $f_0 + \Delta f_0$, *where*

$$\Delta f_0 = \left[R^{-1}R^{-T} - R^{-1}R^{-T}U_k\left(I + U_k^T R^{-1}R^{-T}U_k\right)^{-1}U_k^T R^{-1}R^{-T}\right]q_k,$$

$$U_k = N^T e_k \delta_k \qquad ((n-m) \times n_k \text{ matrix}),$$

$$I = n_k \times n_k \text{ identity matrix},$$

$$q_k = -N^T e_k \delta_k \delta_k^T e_k^T (Nf_0 + f_p).$$

Although orthogonal factorization is numerically superior to the normal equations, poor results for (5) maybe obtained with either method when the element flexibility matrix A is ill conditioned. In a series of papers [14, 15, 12] Paige has developed a scheme which can considerably reduce this difficulty. This gives us motivation to handle reanalysis based on Paige's formulations. Furthermore we apply the perturbation analysis discussed in [14] to the more general case that almost all components of the stiffness matrix have been modified (e.g., perturbation occurs due to excitation of frequency in forced response of an elastic structure to a time-harmonic load).

3. REANALYSIS WITH LEAST-SQUARES SCHEME

In this section we present formulations of Paige's linearly constrained sum-of-squares scheme, and then apply the perturbation analysis with these schemes to our discussion of reanalysis.

3.1. *Paige's Formulation*

Following Paige [14], if we define the weighted residual vector

$$v = G^{-1}(Nf_0 + f_p),$$

then the problem (5) can be written in the equivalent form

$$\min_{v, f_0} v^T v \qquad \text{subject to} \quad Gv = Nf_0 + f_p. \tag{8}$$

Because of its special form, the problem (8) is sometimes referred to as the linearly constrained sum-of-squares problem. In addition to leading to a better numerical method, (8) also has important theoretical advantages over (5) in that it requires no restrictive assumptions regarding the ranks of the matrices involved. In particular, it is possible to compute a G which is suitable for use in (8) even if the element stiffness matrix is only semidefinite. Furthermore (8) could be expressed as

$$\operatorname*{Min}_{v, f_0} \left\| [0, \; I] \begin{bmatrix} f_0 \\ v \end{bmatrix} \right\|_2 \qquad \text{subject to} \quad [-N, \; G] \begin{bmatrix} f_0 \\ v \end{bmatrix} = f_p,$$

a simple equality-constrained least-squares problem. One of the general methods in [13] could then be applied. The method in [13] appears to be the most numerically reliable of these, although no rounding-error analysis is given. However, such a method does not treat f_0, v, N, G separately. In reanalysis of the damaged-structure case, the nullspace basis matrix N and particular solution f_p remain unchanged while G has been modified. So it is important in the analysis to treat them separately. Paige [14] gave a numerically stable algorithm that takes advantage of the special form of (8), and maintains f_0, v, N, G as separate throughout. This will allow us to carry out a reanalysis based on the resulting decomposition. For problems with special structure, as for example in [11], it is also important to maintain f_0, v, N, G as separate during the computation.

FORMULATION I. First, decompose N in (8) as

$$Q^T N = \begin{bmatrix} Q_1^T N \\ Q_2^T N \end{bmatrix} = \begin{bmatrix} R \\ 0 \end{bmatrix}, \tag{9}$$

where R is a nonsingular upper triangular matrix of order $n - m$, $Q = (Q_1, Q_2)$ is an orthogonal matrix of order n, and Q_1 and Q_2 are $n \times (n - m)$ and $n \times m$ matrices, respectively. The constraints in (8) then split into

$$Q_1^T G v = R f_0 + Q_1^T f_p, \tag{10}$$

$$Q_2^T G v = Q_2^T f_p. \tag{11}$$

Since R has full row rank, (10) can always be solved for f_0 once v is given, and so (11) gives the constraints on v, and (8) becomes

$$\operatorname*{Min}_{v} v^T v \qquad \text{subject to} \quad Q_2^T G v = Q_2^T f_p. \tag{12}$$

Next, apply the QR factorization to $(Q_2^T G)^T$ starting from the lower right components to decompose $Q_2^T G$ so that

$$Q_2^T G P = (0, L_2), \tag{13}$$

where $P = (P_1, P_2)$ is an orthogonal matrix of order n, and P_1 and P_2 are $n \times (n - m)$ and $n \times m$ matrices, respectively. Here L_2 has full column rank. That is, decompose $Q^T G$ as

$$Q^T G P = \begin{bmatrix} Q_1^T G P_1 & Q_1^T G P_2 \\ 0 & L_2 \end{bmatrix} = \begin{bmatrix} L_1 & L_{12} \\ 0 & L_2 \end{bmatrix}. \tag{14}$$

Assuming L_2 is nonsingular, we now obtain

$$v = P_2 L_2^{-1} Q_2^T f_p, \tag{15}$$

since $Q_2^T G P_2 = L_2$. Finally, f_0 is recovered from the triangular system (10).

FORMULATION II. Formulation I does not take advantage of any special structure the matrix G may have (G will be triangular if it is computed by the Cholesky factorization, and in our case G has block-diagonal structures as well); indeed, that structure is in general destroyed by the first orthogonal transformation Q. Kourouklis and Paige [12] have given a version of Formulation II in which the two orthogonal transformations U and V (to the corresponding matrices Q and P in (14), respectively) are modified simultaneously in a manner which retains the triangular structure of G throughout the computations. For implementation details, see [15].

The result is a factorization of the form

$$U^T \left[f_p, N, GV \right] = \begin{bmatrix} f_{p1} & 0 & M_1 & 0 \\ f_{p2} & S^T & M_{21} & M_2 \end{bmatrix}, \tag{16}$$

where S^T, M_1, and M_2 are lower triangular matrices of order $n - m$, m, and $n - m$, respectively, and $U = (U_1, U_2)$ and $V = (V_1, V_2)$ are orthogonal matrices of order n and m, respectively. The matrices U_1 and V_1 are $n \times m$, and the matrices U_2 and V_2 are $n \times (n - m)$. We note that the matrices in (16) are not necessarily identical to the corresponding matrices in (14). Applying the transformation (16) and using the change of variable

$$V^T v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}, \tag{17}$$

the problem (8) becomes

$$\underset{v_1, v_2}{\mathrm{Min}}\, v_1^T v_1 + v_2^T v_2 \qquad \text{subject to} \quad \begin{bmatrix} M_1 & 0 \\ M_{21} & M_2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ S^T \end{bmatrix} f_0 + \begin{bmatrix} f_{p1} \\ f_{p2} \end{bmatrix}.$$

Thus v_1 is completely determined by the equation

$$M_1 v_1 = f_{p1}, \tag{18}$$

and the functional is minimized by taking $v_2 = 0$. Finally, the solution f_0 may now be determined from the system

$$S^T f_0 = M_{21} v_1 - f_{p2}. \tag{19}$$

3.2. Reanalysis with Formulation I

The application of Paige's Formulation I to reanalysis can be stated as follows: Suppose we have a Cholesky factor $\bar{G}$ of perturbed element stiffness matrix $\bar{A}^{-1}$, and let our perturbed data result in $\bar{G} = G + \delta G$, leading to the solution $v + \delta v$, $f_0 + \delta f_0$ of the perturbed problem (8). Note that N and f_p remain unchanged in our approach to reanalyzing structures using the force method. Considering (8) for both the original and perturbed problems, we see that δv and δf_0 give

$$\underset{\delta v, \delta f_0}{\mathrm{Min}}\ 2v^T(\delta v) + (\delta v)^T(\delta v) \tag{20}$$

subject to

$$\bar{G}(\delta v) = N(\delta f_0) - (\delta G)v. \tag{21}$$

The constraints (21) have the same form as in (8), so we can proceed as in (14):

$$Q^T \bar{G} \bar{P} = \begin{bmatrix} \bar{L}_1 & \bar{L}_{12} \\ 0 & \bar{L}_2 \end{bmatrix}, \tag{22}$$

where $\bar{P} = (\bar{P}_1, \bar{P}_2)$ is orthogonal, and $\bar{L}_2$ has full column rank. For solving a sequence of problems with fixed N but varying G, in theory it is necessary to compute the orthogonal transformation Q only once. Applying the perturbation analysis discussed in Paige [14] to our case, we get the following results, which could be viewed as an important special case of Paige's work.

LEMMA 1. *Let v be the solution of* (12). *Let G be the Cholesky factor of the symmetric positive definite matrix A^{-1}, and N has full column rank. Assume L_2 and $\bar{L}_2$ are nonsingular. If δv satisfies* (20) *and* (21), *then*

$$\delta v = \left[\bar{P}_1 \bar{P}_1^T (\delta G)^T Q_2 (L_2^{-1})^T P_2^T + \bar{P}_2 \bar{L}_2^{-1} Q_2^T (\delta G)\right] v \tag{23}$$

and

$$\|\delta v\|_2 \leqslant \left[\frac{1}{\sigma(L_2)} + \frac{1}{\sigma(\bar{L}_2)}\right] \epsilon_G \|v\|_2 \equiv \beta_v, \tag{24}$$

where $\epsilon_G = \|\delta G\|_2$, Q_2 is as in (9), *and $\sigma(L_2)$ and $\sigma(\bar{L}_2)$ are the smallest nonzero singular values of L_2 and $\bar{L}_2$, respectively.*

Proof. From combining (21) and (22), that is,

$$\begin{bmatrix} \bar{L}_1 & \bar{L}_{12} \\ 0 & \bar{L}_2 \end{bmatrix} \begin{bmatrix} \bar{P}_1^T \\ \bar{P}_2^T \end{bmatrix} (\delta v) = \begin{bmatrix} R \\ 0 \end{bmatrix} (\delta f_0) - \begin{bmatrix} Q_1^T \\ Q_2^T \end{bmatrix} (\delta G) v,$$

we get

$$\bar{L}_2 \bar{P}_2^T (\delta v) = -Q_2^T (\delta G) v, \tag{25}$$

and this must be a consistent system for the perturbation to be meaningful for this problem, since (25) is an underdetermined system. We can then express

$$\delta v = \bar{P}_1 z_1 + \bar{P}_2 z_2, \quad z_2 = \bar{L}_2^{-1} Q_2^T (\delta G) v \qquad \text{for all } z_1, \tag{26}$$

since $\bar{L}_2 \bar{P}_2^T = Q_2^T \bar{G}$ and $Q_2^T \bar{G} \bar{P}_1 = 0$. Substituting (26) in (20) and taking the derivative with respect to z_1 gives

$$z_1 = -\bar{P}_1^T v, \qquad \delta v = -\bar{P}_1 \bar{P}_1^T v + \bar{P}_2 \bar{L}_2^{-1} Q_2^T (\delta G) v. \tag{27}$$

The second term of (27) can easily be bounded, but the first is difficult to bound. From (15),

$$\bar{P}_1^T v = \bar{P}_1^T P_2 L_2^{-1} Q_2^T f_p = \bar{P}_1^T P_2 P_2^T v, \tag{28}$$

and we will seek an expression for $\bar{P}_1^T P_2$. To do this we first consider the following expression from (22):

$$Q^T (G + \delta G) \bar{P} = \begin{bmatrix} \bar{L}_1 & \bar{L}_{12} \\ 0 & \bar{L}_2 \end{bmatrix}.$$

Then we compare the first set of columns of both sides, obtaining

$$Q_2^T (\delta G) \bar{P}_1 = -Q_2^T G \bar{P}_1 = -L_2 P_2^T \bar{P}_1,$$

since $Q_2^T G P_2 = L_2$. This can be used with (27) and (28) to give an expression for δv as follows:

$$\begin{aligned}
\delta v &= -\bar{P}_1 \bar{P}_1^T v + \bar{P}_2 \bar{L}_2^{-1} (\delta G) v \\
&= -\bar{P}_1 \left(\bar{P}_1^T P_2 P_2^T v \right) + \bar{P}_2 \bar{L}_2^{-1} Q_2^T (\delta G) v \\
&= -\bar{P}_1 \left[L_2^{-1} \left(-Q_2^T (\delta G) \bar{P}_1 \right) \right]^T P_2^T v + \bar{P}_2 \bar{L}_2^{-1} Q_2^T (\delta G) v \\
&= \bar{P}_1 \bar{P}_1^T (\delta G)^T Q_2 L_2^{-T} P_2^T v + \bar{P}_2 \bar{L}_2^{-1} Q_2^T (\delta G) v.
\end{aligned}$$

By taking the 2-norm we obtain

$$\begin{aligned}
\|\delta v\| &= \left[\left\| \bar{P}_1 \bar{P}_1^T (\delta G)^T Q_2 L_2^{-T} P_2^T \right\| + \left\| \bar{P}_2 \bar{L}_2^{-1} Q_2^T (\delta G) \right\| \right] \|v\| \\
&\leqslant \left[\left\| (\delta G)^T \right\| \|L_2^{-T}\| + \|\bar{L}_2^{-1}\| \, \|\delta G\| \right] \|v\|.
\end{aligned}$$

Let $\epsilon_G = \|\delta G\|$, and let $\sigma(L_2)$ and $\sigma(\bar{L}_2)$ be the smallest nonzero singular value of L_2 and $\bar{L}_2$ respectively, then

$$\|\delta v\| \leqslant \left[\frac{\epsilon_G}{\sigma(L_2)} + \frac{\epsilon_G}{\sigma(\bar{L}_2)} \right] \|v\|. \qquad \blacksquare$$

THEOREM 2. *Let f_0 be the solution of* (10), *that is, the solution of* (4). *Let G be the Cholesky factor of the symmetric positive definite matrix A^{-1}, and N has full column rank. Then the solution of* (20) *is given by $f_0 + \delta f_0$, where*

$$\delta f_0 = R^{-1}\Big[\big(\bar{L}_{12}\bar{L}_2^{-1}Q_2^T + Q_1^T\big)(\delta G) + \bar{L}_1\bar{P}_1^T(\delta G)^T Q_2 L_2^{-T}P_2^T\Big]v \quad (29)$$

and

$$\|\delta f_0\|_2 \leqslant \left[\left(1 + \frac{\|\bar{L}_{12}\|_2}{\sigma(\bar{L}_2)}\right) + \left(\frac{\|\bar{L}_1\|_2}{\sigma(L_2)}\right)\right]\frac{\epsilon_G\|v\|_2}{\sigma(N)} \equiv \beta_{f_0}; \quad (30)$$

$\epsilon_G = \|\delta G\|_2$, *and Q is as in* (9); *and $\sigma(L_2)$, $\sigma(\bar{L}_2)$, and $\sigma(N)$ are the smallest nonzero singular values of L_2, $\bar{L}_2$, and N, respectively.*

Proof. From combining (9) and (21),

$$\begin{aligned}
&Q^T N(\delta f_0) - Q^T\bar{G}(\delta v) = Q^T(\delta G)v \\
\Rightarrow\quad &\begin{bmatrix} R \\ 0 \end{bmatrix}(\delta f_0) - \begin{bmatrix} Q_1^T \\ Q_2^T \end{bmatrix}\bar{G}(\delta v) = \begin{bmatrix} Q_1^T \\ Q_2^T \end{bmatrix}(\delta G)v \\
\Rightarrow\quad &R(\delta f_0) - Q_1^T\bar{G}(\delta v) = Q_1^T(\delta G)v \\
\Rightarrow\quad &\delta f_0 = R^{-1}\big[Q_1^T(\delta G)v + Q_1^T\bar{G}(\delta v)\big],
\end{aligned}$$

since N has full column rank. By using (23) in the previous lemma, and since $Q_1^T\bar{G}\bar{P}_2 = \bar{L}_{12}$ and $Q_1^T\bar{G}\bar{P}_1 = \bar{L}_1$,

$$\begin{aligned}
\delta f_0 &= R^{-1}\Big\{Q_1^T(\delta G)v \\
&\qquad + Q_1^T\bar{G}\Big[+\bar{P}_2\bar{L}_2^{-1}Q_2^T(\delta G)v + \bar{P}_1\bar{P}_1^T(\delta G)^T Q_2 L_2^{-T}P_2^T v\Big]\Big\} \\
&= R^{-1}\Big[Q_1^T(\delta G)v + \bar{L}_{12}\bar{L}_2^{-1}Q_2^T(\delta G)v + \bar{L}_1\bar{P}_1^T(\delta G)^T Q_2 L_2^{-T}P_2^T v\Big] \\
&= R^{-1}\Big[\big(\bar{L}_{12}\bar{L}_2^{-1}Q_2^T + Q_1^T\big)(\delta G) + \bar{L}_1\bar{P}_1^T(\delta G)^T Q_2 L_2^{-T}P_2^T\Big]v.
\end{aligned}$$

Since $\sigma(R) = \sigma(N)$, we can get (30) from (29) by taking the 2-norm. ■

Note that if we assume L_2 and $\overline{L}_2$ are nonsingular, that is, $Q_2^T G$ and $Q_2^T \overline{G}$ have full row rank, then the computation for δf_0 is simple. Based on various assumptions (Paige [14]), we can get a tighter bound for δf_0 than (30).

3.3. Reanalysis with Formulation II

Applying Paige's Formulation II to reanalysis can proceed as in (16) for the perturbed problem (20):

$$U^T\left[f_p, N, \overline{G}\overline{V}\right] = \begin{bmatrix} f_{p1} & 0 & \overline{M}_1 & 0 \\ f_{p2} & S^T & \overline{M}_{21} & \overline{M}_2 \end{bmatrix}, \tag{31}$$

where $\overline{V} = (\overline{V}_1, \overline{V}_2)$ is orthogonal, and $\overline{M}_1$ has full column rank. Again, for reanalysis with fixed N but varying G, in theory it is necessary to compute the orthogonal transformation U only once. With similar proof to Lemma 1, we get following result for reanalysis with formulation II.

COROLLARY 1. *Let v be the solution of* (18). *Let G be the Cholesky factor of the symmetric positive definite matrix A^{-1}, and N has full column rank. Assume M_1 and $\overline{M}_1$ are nonsingular. If δv satisfies* (20) *and* (21), *then*

$$\delta v = \left[\overline{V}_2 \overline{V}_2^T (\delta G)^T U_1 (M_1^{-1})^T V_1^T + \overline{V}_1 \overline{M}_1^{-1} U_1^T (\delta G)\right] v \tag{32}$$

and

$$\|\delta v\|_2 \leqslant \left[\frac{1}{\sigma(M_1)} + \frac{1}{\sigma(\overline{M}_1)}\right] \epsilon_G \|v\|_2, \tag{33}$$

where $\epsilon_G = \|\delta G\|_2$ and U_2 is as in (16), *and $\sigma(M_1)$ and $\sigma(\overline{M}_1)$ are the smallest nonzero singular values of M_1 and $\overline{M}_1$, respectively.*

Proof. From combining (21) and (31)

$$\begin{bmatrix} \overline{M}_1 & 0 \\ \overline{M}_{21} & \overline{M}_2 \end{bmatrix} \begin{bmatrix} \overline{V}_1^T \\ \overline{V}_2^T \end{bmatrix} (\delta v) = \begin{bmatrix} 0 \\ S^T \end{bmatrix} (\delta f_0) - \begin{bmatrix} U_1^T \\ U_2^T \end{bmatrix} (\delta G) v,$$

we get

$$\overline{M}_1\overline{V}_1^T(\delta v) = -U_1^T(\delta G)v,$$

and then express

$$\delta v = \overline{V}_2 z_2 + \overline{V}_1 z_1, \quad z_1 = \overline{M}_1^{-1}U_1^T(\delta G)v \qquad \text{for all } z_2, \tag{34}$$

since $\overline{M}_1\overline{V}_1^T = U_1^T\overline{G}$ and $U_1^T\overline{G}\overline{V}_2 = 0$. Substituting (34) in (20) and taking the derivative with respect to z_2 gives

$$z_2 = -\overline{V}_2^T v, \qquad \delta v = -\overline{V}_2\overline{V}_2^T v + \overline{V}_1\overline{M}_1^{-1}U_1^T(\delta G)v. \tag{35}$$

By using (17) and (18), the equations (35) can be expressed as

$$z_2 = -\overline{V}_2^T v, \qquad \delta v = -\overline{V}_2\overline{V}_2^T V_1 v_1 + \overline{V}_1\overline{M}_1^{-1}U_1^T(\delta G)V_1 v_1. \tag{36}$$

In a similar manner to the proof of Lemma 1, we will seek an expression for $\overline{V}_2^T V_1$. We first consider the following expression:

$$U^T(G + \delta G)\overline{V} = \begin{bmatrix} \overline{M}_1 & 0 \\ \overline{M}_{21} & \overline{M}_2 \end{bmatrix}.$$

Then we get the following, by comparing the second set of columns on the two sides:

$$U_1^T(\delta G)\overline{V}_2 = -U_1^T G\overline{V}_2 = -M_1 V_1^T\overline{V}_2,$$

since $U_1^T(G + \delta G)\overline{V}_2 = 0$ and $U_1^T G V_1 = M_1$. This can be used with (36) to gives an expression for δv as follows:

$$\delta v = \overline{V}_2\overline{V}_2^T(\delta G)^T U_1 M_1^{-T} v_1 + \overline{V}_1\overline{M}_1^{-1}U_1^T(\delta G)V_1 v_1.$$

Taking the 2-norm, we obtain

$$\begin{aligned} \|\delta v\| &= \left[\left\|\overline{V}_2\overline{V}_2^T(\delta G)^T U_1 M_2^{-T}\right\| + \left\|\overline{V}_1\overline{M}_1^{-1}U_1^T(\delta G)V_1\right\|\right]\|v_1\| \\ &\leqslant \left[\left\|(\delta G)^T\right\|\|M_1^{-T}\| + \|\overline{M}_1^{-1}\|\,\|\delta G\|\right]\|v_1\|. \end{aligned}$$

Let $\sigma(M_1)$ and $\sigma(\overline{M}_1)$ be the smallest nonzero singular values of M_1 and $\overline{M}_1$ respectively. Then since $\|v\| = \|v_1\|$, we have

$$\|\delta v\| \leqslant \left[\frac{\epsilon_G}{\sigma(M_1)} + \frac{\epsilon_G}{\sigma(\overline{M}_1)}\right]\|v\|. \qquad \blacksquare$$

COROLLARY 2. *Let f_0 be the solution of* (19), *that is, the solution of* (4). *Let G be the Cholesky factor of the symmetric positive definite matrix A^{-1}, and N has full column rank. Then the solution of* (20) *is given by $f_0 + \delta f_0$, where*

$$\delta f_0 = S^{-1}\left[\left(\overline{M}_{21}\overline{M}_1^{-1}U_1^T + U_2^T\right)(\delta G) + \overline{M}_2\overline{V}_2^T(\delta G)^T U_1 M_1^{-T}V_1^T\right]v \quad (37)$$

and

$$\|\delta f_0\|_2 \leqslant \left[\left(1 + \frac{\|\overline{M}_{21}\|_2}{\sigma(\overline{M}_1)}\right) + \left(\frac{\|\overline{M}_2\|_2}{\sigma(M_1)}\right)\right]\frac{\epsilon_G\|v\|_2}{\sigma(N)}; \quad (38)$$

$\epsilon_G = \|\delta G\|_2$ *and U is as in* (16); *and $\sigma(M_1)$, $\sigma(\overline{M}_1)$, and $\sigma(N)$ are the smallest nonzero singular values of M_1, $\overline{M}_1$, and N, respectively.*

Proof. From combining (16) and (21),

$$U^T N(\delta f_0) - U^T\overline{G}(\delta v) = U^T(\delta G)v$$

$$\Rightarrow \quad \delta f_0 = S^{-1}\left[U_2^T(\delta G)v + U_2^T\overline{G}(\delta v)\right],$$

since N has full column rank. By using (32) in Corollary 1, since $U_2^T\overline{G}\overline{V}_1 = \overline{M}_{21}$ and $U_2^T\overline{G}\overline{V}_2 = \overline{M}_2$,

$$\begin{aligned}\delta f_0 &= S^{-1}\Big\{U_2^T(\delta G)v \\ &\qquad + U_2^T\overline{G}\left[\overline{V}_1\overline{M}_1^{-1}U_1^T(\delta G)v + \overline{V}_2\overline{V}_2^T(\delta G)^T U_1 M_1^{-T}V_1^T v\right]\Big\} \\ &= S^{-1}\left[\left(\overline{M}_{21}\overline{M}_1^{-1}U_1^T + U_2^T\right)(\delta G) + \overline{M}_2\overline{V}_2^T(\delta G)^T U_1 M_1^{-T}V_1^T\right]v.\end{aligned}$$

Since $\sigma(S) = \sigma(N)$, we can get (38) from (37) by taking the 2-norm. $\blacksquare$

3.4. Numerical Experiments

In this section we present some results of numerical tests.

EXAMPLE. Consider the two-dimensional frame with 15 elements and 9 nodes which is shown in Figure 1. In this case the equilibrium matrix is 27×45 and the element flexibility matrix is 45×45 matrix.

Damage will be measured on an element-by-element basis, and will consist of two separate measurements: d_K^i, the "flexibility damage" to the ith element, and d_M^i, the "mass damage." The numbers d_K^i and d_M^i lie between 0 and 1 inclusive, and represent a fractional decrease in load capacity and mass, respectively. Generally, $d_M^i = 0$ except in cases of physical removal of the element, in which case $d_M^i = 1$. It is conceivable that, if the damage is a hole in the interior of an element, d_M^i may lie strictly between 0 and 1, but the value of d_M^i will not affect the method of analysis. d_K^i may have any value in the range stated. The effect upon element flexibility is thus $[A_i]_{dF} = [A_i]/(1 - d_K^i)$, where dF refers to values in the damaged state and $[A_i]$ is the ith block (corresponding to the ith element) of the flexibility matrix. Some numerical results obtained by using MATLAB (with various d_K^i and corresponding condition number $\kappa_2(A)$) are listed in Tables 1 and 2. Here, β_v and β_{f_0} are from Lemma 1 and Theorem 1, respectively.

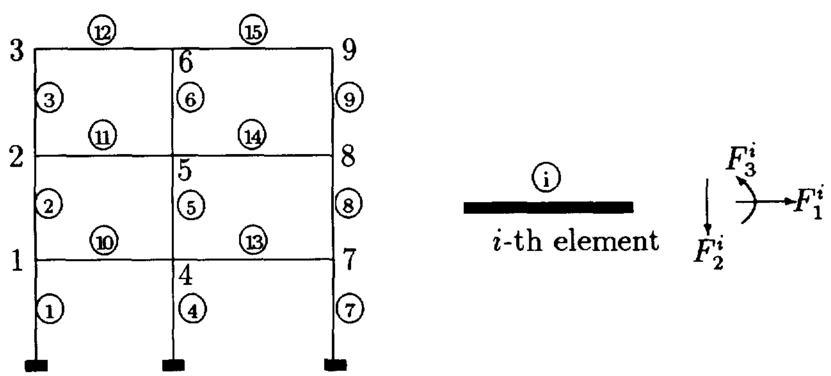

FIG. 1. Two-dimensional frame with element and node numbering.

TABLE 1

NUMERICAL RESULTS: ONE ELEMENT (A_i, FOR $i = 12$) MODIFIED

d_K^{12}	$\kappa_2(A)$	$\|\delta v\|_2$	β_v	$\|\delta f_0\|_2$	β_{f_0}
0.99999	8.69e + 08	2.30e − 01	1.62e + 02	2.95e + 01	1.14e + 05
0.99	8.68e + 05	2.07e − 01	1.47e + 02	2.90e + 01	1.03e + 05
0.9	8.68e + 04	1.53e − 01	1.10e + 02	2.47e + 01	7.59e + 04
0.09	9.55e + 03	8.20e − 03	7.14e + 00	1.60e + 00	4.31e + 03
0.0009	8.70e + 03	7.85e − 05	6.96e − 02	1.53e − 02	4.16e + 01
0.000009	8.69e + 03	7.85e − 07	6.96e − 04	1.54e − 04	4.15e − 01

The bounds on v and f_0 that we get from Corollary 1 and Corollary 2 are the same as the results in Tables 1 and 2 in our experiments. From the way we obtained $\overline{L}_2$ (in (22)) and $\overline{M}_1$ (in (31)), it is natural that we should get the same smallest nonzero singular values $\sigma(\overline{L}_2)$ and $\sigma(\overline{M}_1)$ from both formulations. The same argument can be applied to $\sigma(L_2)$ and $\sigma(M_1)$ obviously. However, the matrices U and V in (16) are not necessarily identical to the corresponding matrices in (14), as we have already stated.

4. CONCLUSIONS

We have established the boundedness of f_0 in (4) when a perturbation occurs on the element flexibility matrix. The analysis and subsequent reanalysis of structures were performed by using the force method of analysis. Unlike similar methods that use the displacement method of analysis, the

TABLE 2

NUMERICAL RESULTS: FIVE ELEMENTS (A_i, FOR $i = 1, 2, 3, 13, 14$) MODIFIED

$d_K^{1,2,3}$	$d_K^{13,14}$	$\kappa_2(A)$	$\|\delta v\|_2$	β_v	$\|\delta f_0\|_2$	β_{f_0}
0.99999	0.000009	2.58e + 07	2.93e + 00	3.72e + 02	2.02e + 02	2.76e + 05
0.99	0.0009	2.57e + 04	2.64e + 00	3.35e + 02	1.98e + 02	2.45e + 05
0.9	0.09	8.69e + 03	1.96e + 00	2.51e + 02	1.71e + 02	1.67e + 05
0.09	0.9	1.23e + 04	9.16e − 01	2.08e + 02	1.36e + 02	1.32e + 05
0.0009	0.99	1.23e + 05	1.41e + 00	2.86e + 02	1.97e + 02	1.93e + 05
0.000009	0.99999	1.23e + 08	1.60e + 00	3.19e + 02	2.07e + 02	2.17e + 05

procedure presented here yields a direct, rather than an iterative, method, which is an important consideration when analyzing large-scale structural systems [3].

Numerical results are presented for a two-dimensional frame. For the element and node numbering for this structure example, we used the method in [16], which we call the substructuring method with proper partitions. Retaining the triangular structure of G and the special structure of N from the substructuring method enables us to reduce the amount of work throughout the computations. On the basis of our numerical results we may be able to get a tighter bound for δf_0.

REFERENCES

1 J. H. Argyris and S. Kelsey, *Energy Theorems and Structural Analysis*, Butterworth, London, 1960.

2 J. S. Arora, D. F. Haskell, and A. K. Gavil, Optimal design of large structures for damage tolerances, *AIAA J.* 18 (1980).

3 J. R. Batt and S. Gellin, Rapid reanalysis by the force method, *Comput. Appl. Mech. and Engrg.* 53:105–117 (1985).

4 J. R. Batt, S. Gellin, and R. A. Gellatly, Force Method Optimization II: Volume I, Theoretical manual, AFWAL TR-82-3088, Dec. 1982.

5 M. Berry, M. Heath, I. Kaneko, M. Lawo, R. Plemmons, and R. Ward, An algorithm to compute a sparse basis of the null space, *Numer. Math.* 47:483–504 (1985).

6 T. F. Coleman and A. Pothen, The null space problem I. Complexity, *SIAM J. Algebraic Discrete Methods* 7(4):527–537 (1986).

7 T. F. Coleman and A. Pothen, The null space problem II. Complexity, *SIAM J. Algebraic Discrete Methods* 8(4):544–563 (1987).

8 J. R. Gilbert and M. T. Heath, Computing a sparse basis for the null space, *SIAM J. Algebraic Discrete Methods* 8(3):446–459 (1987).

9 M. Heath, R. Plemmons, and R. Ward, Sparse orthogonal schemes for structural optimization using the force method, *SIAM J. Sci. Statist. Comput.* 5(3):514–532 (1984).

10 F. D. Hemming and V. B. Venkayya, Efficiency Considerations in Flutter Optimization with Effects of Damage Included, AIAA Paper 80-0788-CP, presented at 21st AIAA/ASME/ASCE/AHS Conference, Seattle, May 1980.

11 H. Jang, Equilibrium Equation: Substructuring and the Force Method, Ph.D. Thesis, North Carolina State Univ., 1991.

12 S. Kourouklis and C. C. Paige, A constrained least squares approach to the general Gauss-Markov linear model, *J. Amer. Statist. Assoc.* 76:620–625 (1981).

13 C. L. Lawson and R. J. Hanson, *Solving Least Squares Problems*, Prentice-Hall, Englewood Cliffs, N.J., 1974.

14 C. C. Paige, Computer solution and perturbation analysis of generalized linear least squares problems, *Math. Comput.* 33:171–183 (1979).

15 C. C. Paige, Fast numerically stable computations for generalized linear least squares problems, *SIAM J. Numer. Anal.* 16:165–171 (1979).

16 R. Plemmons and R. White, Substructuring methods for computing the nullspace of equilibrium matrices, *SIAM J. Matrix Anal. Appl.* 11(1):1–22 (1990).

17 D. S. Scott, J. C. Westkaemper, A. Sejal, and R. O. Stearman, The Influence of Ballistic Damage on the Aeroelastic Characteristics of Lifting Surfaces, AFOSR TR-80-0220, 1979.

18 University of Dayton Research Institute, Optimization and Damage Assessment of Aerospace Structures, USAF Contract F33615-79-C-3209.

Received 15 March 1996; final manuscript accepted 29 October 1996